

Modular classes of skew algebroid relations *

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Abstract

Skew algebroid is a natural generalization of the concept of Lie algebroid. In this paper, for a skew algebroid E , its *modular class* $\text{mod}(E)$ is defined in the classical as well as in the supergeometric formulation. It is proved that there is a homogeneous nowhere-vanishing 1-density on E^* which is invariant with respect to all Hamiltonian vector fields if and only if E is modular, i.e. $\text{mod}(E) = 0$. Further, *relative modular class* of a subalgebroid is introduced and studied together with its application to holonomy, as well as *modular class of a skew algebroid relation*. These notions provide, in particular, a unified approach to the concepts of a modular class of a Lie algebroid morphism and that of a Poisson map.

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1 Introduction

The concept of *modular class* of a Lie algebroid, proposed by Weinstein and introduced in [6, 41], is a natural extension of that for a Poisson manifold [28, 41] and hence an analogue of the modular automorphism group of a von Neumann algebra. It is well known that a Lie algebroid structure on a vector bundle E is canonically associated with a linear Poisson structure on the dual bundle E^* , and the modular class of E can be viewed as the obstruction to the existence of a homogeneous measure on E^* being invariant with respect to all Hamiltonian vector fields on the Poisson manifold E^* [41]. The question of the existence of a measure on E^* invariant with respect to certain Hamiltonian vector fields of mechanical origins was studied recently by Marrero [32]. In the framework of Lie-Rinehart algebras, the concept of modular class was developed by Huebschmann [21], and the role of Batalin-Vilkovisky algebras was emphasized in [23, 42].

It was then observed in [15] that the modular class can be naturally defined also for a base-preserving morphism of Lie algebroids. A similar object has been studied in [26] under the name *relative modular class* and then extended to arbitrary Lie algebroid morphisms [27]. In [8] Fernandes constructed a sequence of secondary characteristic classes of a Lie algebroid

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whose first element coincides with the modular class (see also [29]). Secondary characteristic classes of a base-preserving Lie algebroid morphism were studied also by Vaisman [37].

Poisson manifolds were revisited recently by Caseiro and Fernandes in [2], where modular class of a Poisson map was defined and studied in detail. The problem was that, although any Poisson structure Λ on a manifold M gives rise to a Lie algebroid structure on the cotangent bundle T^*M and the modular class of this Lie algebroid is just twice the modular class of the Poisson manifold, a Poisson map $\varphi : M_1 \rightarrow M_2$ does not in general induce any canonical Lie algebroid morphism between T^*M_1 and T^*M_2 .

On the other hand, several concepts of objects generalizing Lie algebroids appeared recently in the literature. One of them is the concept of *general algebroid* or its skew-symmetric version, a *skew algebroid*, introduced in [17, 18] and applied in analytical mechanics in [10, 12] as an extension of Lie algebroid/groupoid description of Lagrangian and Hamiltonian formalisms proposed by Weinstein [40]. The interest to skew algebroids, for which the Jacobi identity valid for Lie algebroids is dropped, comes from the nonholonomic mechanics in which they provide a natural geometric framework (see [14] and references therein).

Our aim in this paper is twofold: first, to define modular class for a skew algebroid and, second, to extend the concept of modular class of a Lie algebroid morphism to that of a linear relation between skew algebroids. The idea is very simple and comes from considering Poisson maps. Namely, if $\varphi : M_1 \rightarrow M_2$ is a Poisson map between Poisson manifolds, then the tangent bundle $T\text{graph}(\varphi)$ of the graph of φ is a coisotropic subbundle in the direct product $TM_1 \times \overline{TM}_2 \simeq T(M_1 \times \overline{M}_2)$, where \overline{M} denotes the manifold M with the opposite Poisson structure. Hence, its annihilator $\mathcal{R} = (T\text{graph}(\varphi))^\perp$ is a Lie subalgebroid in the direct product Lie algebroid $T^*M_1 \times \overline{T^*M}_2$, thus, by definition, a *Lie algebroid relation*. Hence, dealing with a Poisson map is just a particular case of dealing with a Lie algebroid relation for which modular class can be naturally defined, and the whole picture can be easily extended to the context of skew algebroids.

Note that this framework is not only unifying several concepts of modular class and generalizes the setting by dropping the Jacobi identity, but it fits also much better to the modern approach to various problems in geometry by considering relations rather than maps.

The paper is organized as follows. In Section 2 we present basic notions and facts from the theory of skew algebroids. Modular class of a skew algebroid and characterizations of modularity are presented in Section 3 together with an application to nonholonomic mechanics. Section 4 contains an elementary proof that the modular class can be simply defined in the language of supergeometry as the (super)divergence of the de Rham vector field (homological vector field in the case of a Lie algebroid) with respect to a natural class of Berezinian non-oriented densities on the supermanifold $E[1]$. Relative modular class of a subalgebroid is defined in Section 5 and then used in Section 6 to a nice description of the corresponding holonomy along an admissible path. Finally, in Section 7, we present the modular class of a skew algebroid relation (and, more generally, a linear relation between skew algebroids) together with showing its unifying role in the theory.

2 Skew algebroids

Let $\tau : E \rightarrow M$ be a rank- n vector bundle over an m -dimensional manifold M , and let $\pi : E^* \rightarrow M$ be its dual. Let $\mathcal{A}^i(E) = \text{Sec}(\wedge^i E)$, for $i = 0, 1, 2, \dots$, denotes the module of sections of the bundle $\wedge^i E$. Let us put $\mathcal{A}^i(E) = \{0\}$ for $i < 0$, and denote with $\mathcal{A}(E) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}^i(E)$ the Grassmann algebra of multisections of E . It is a graded commutative associative algebra

with respect to the wedge product. Similarly we define the Grassmann algebra $\mathcal{A}(E^*)$ for the dual bundle E^* .

There are different equivalent ways to define a *skew algebroid* structure on E . Here we will list only three of them. The notation is borrowed from [18, 12, 10] and we refer to these papers for details. In particular, we use affine coordinates (x^a, ξ_i) on E^* and the dual coordinates (x^a, y^i) on E , as associated with dual local bases, (e_i) and (e^i) , of sections of E and E^* , respectively.

Definition 2.1. A *skew algebroid* structure on E is given by a linear bivector field Π on E^* . In local coordinates,

$$\Pi = c_{ij}^k(x) \xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} + \rho_i^b(x) \partial_{\xi_i} \wedge \partial_{x^b}, \quad (1)$$

where $c_{ij}^k(x) = -c_{ji}^k(x)$. The *opposite* algebroid \bar{E} is the skew algebroid corresponding to the bivector field $-\Pi$. If Π is a Poisson tensor, we speak about a *Lie algebroid*.

As the bivector field Π defines a bilinear bracket $\{\cdot, \cdot\}_\Pi$ on the algebra $C^\infty(E^*)$ of smooth functions on E^* by $\{\phi, \psi\}_\Pi = \langle \Pi, d\phi \wedge d\psi \rangle$, where $\langle \cdot, \cdot \rangle$ stands for contractions, we get the following.

Theorem 2.2. A *skew algebroid structure* (E, Π) can be equivalently defined as

- a skew-symmetric \mathbb{R} -bilinear bracket $[\cdot, \cdot]_\Pi$ on the space $\text{Sec}(E)$ of sections of E , together with a vector bundle morphisms $\rho: E \rightarrow TM$ (the anchor), such that

$$[X, fY]_\Pi = \rho(X)(f)Y + f[X, Y]_\Pi, \quad (2)$$

for all $f \in C^\infty(M)$, $X, Y \in \text{Sec}(E)$,

- or as a derivation d^Π of degree 1 in the Grassmann algebra $\mathcal{A}(E^*)$ (the de Rham derivative). The latter is a map $d^\Pi: \mathcal{A}(E^*) \rightarrow \mathcal{A}(E^*)$ such that $d^\Pi: \mathcal{A}^i(E^*) \rightarrow \mathcal{A}^{i+1}(E^*)$ and, for $\alpha \in \mathcal{A}^a(E^*)$, $\beta \in \mathcal{A}^b(E^*)$, we have

$$d^\Pi(\alpha \wedge \beta) = d^\Pi \alpha \wedge \beta + (-1)^a \alpha \wedge d^\Pi \beta. \quad (3)$$

The bracket $[\cdot, \cdot]_\Pi$ and the anchor ρ are related to the bracket $\{\cdot, \cdot\}_\Pi$ according to the formulae

$$\iota([X, Y]_\Pi) = \{\iota(X), \iota(Y)\}_\Pi, \quad (4)$$

$$\pi^*(\rho(X)(f)) = \{\iota(X), \pi^* f\}_\Pi, \quad (5)$$

where we denoted with $\iota(X)$ the linear function on E^* associated with the section X of E , i.e. $\iota(X)(e_p^*) = \langle X(p), e_p^* \rangle$ for each $e_p^* \in E_p^*$.

The *exterior derivative* d^Π is determined by the formula

$$(d^\Pi \mu)^v = [\Pi, \mu^v], \quad (6)$$

where μ^v is the natural vertical lift of a k -form $\mu \in \mathcal{A}^k(E^*)$ to a vertical k -vector field on E^* and $[\cdot, \cdot]$ is the Schouten bracket of multivector fields.

In local bases of sections and the corresponding local coordinates,

$$[e_i, e_j]_\Pi(x) = c_{ij}^k(x) e_k, \quad (7)$$

$$\rho(e_i)(x) = \rho_i^a(x) \partial_{x^a}, \quad (8)$$

$$d^\Pi f(x) = \rho_i^a(x) \frac{\partial f}{\partial x^a}(x) e^i, \quad (9)$$

$$d^\Pi e^i(x) = c_{lk}^i(x) e^k \wedge e^l. \quad (10)$$

Of course, in general $(d^\Pi)^2 \neq 0$, and $(d^\Pi)^2 = 0$ if and only if Π is a Poisson tensor, thus our skew algebroid is actually a Lie algebroid. If the skew algebroid satisfies the identity (e.g., it is a Lie algebroid)

$$\rho([X, X']_\Pi) = [\rho(X), \rho(X')], \quad (11)$$

we speak about an *almost Lie algebroid*. For an almost Lie algebroid, $(d^\Pi)^2 = 0$ as a map from $\mathcal{A}^0(E^*) = C^\infty(M)$ to $\mathcal{A}^2(E^*)$, thus the standard first cohomology space $H^1(E)$ is well defined as well as homotopies of admissible curves (see [13]).

Remark 2.3. (supergeometric picture) Note that the formula (6) is equivalent to the well-known Cartan's formula for the de Rham derivative. Moreover, as the Grassmann algebra $\mathcal{A}(E^*)$ can be understood as the algebra of smooth functions on the graded (super)manifold $E[1]$ (an N -manifold of degree 1 in the terminology of Ševera and Roytenberg [35, 36]), following [38] we can view the de Rham derivative d^Π as a vector field of degree 1 on $E[1]$. This vector field is *homological*, $(d^\Pi)^2 = 0$, if and only if we are actually dealing with a Lie algebroid. In local supercoordinates (x, \mathbf{y}) associated canonically with our standard affine coordinates (x, y) we have

$$d^\Pi = \frac{1}{2} c_{ij}^k(x) \mathbf{y}^j \mathbf{y}^i \partial_{\mathbf{y}^k} + \rho_i^b(x) \mathbf{y}^i \partial_{x^b}. \quad (12)$$

For any section $X \in \text{Sec}(E)$, the *Lie derivative* \mathcal{L}_X^Π , acting in $\mathcal{A}(E)$ and $\mathcal{A}(E^*)$, is defined in the standard way: $\mathcal{L}_X^\Pi(f) = \rho(X)(f)$ for $f \in C^\infty(M)$,

$$\mathcal{L}_X^\Pi(Y_1 \wedge \cdots \wedge Y_a) = \sum_i Y_1 \wedge \cdots \wedge [X, Y_i]_\Pi \wedge \cdots \wedge Y_a$$

for $Y_1 \wedge \cdots \wedge Y_a \in \mathcal{A}^a(E)$, and

$$\langle \mathcal{L}_X^\Pi(\alpha), Y_1 \wedge \cdots \wedge Y_a \rangle = \mathcal{L}_X^\Pi \langle \alpha, Y_1 \wedge \cdots \wedge Y_a \rangle - \langle \alpha, \mathcal{L}_X^\Pi(Y_1 \wedge \cdots \wedge Y_a) \rangle, \quad (13)$$

for $\alpha \in \mathcal{A}^a(E^*)$, $a > 0$. The formula (13) is equivalent to the Cartan's formula $\mathcal{L}_X^\Pi = i_X d^\Pi + d^\Pi i_X$.

Recall that, given a skew algebroid E , we can associate with any C^1 -function H on E^* its *Hamiltonian vector field* \mathfrak{X}_H like in the standard case: $\mathfrak{X}_H = i_{dH} \Pi$. In local coordinates,

$$\mathfrak{X}_H(x, \xi) = \left(c_{ij}^k(x) \xi_k \frac{\partial H}{\partial \xi_i}(x, \xi) - \rho_j^a(x) \frac{\partial H}{\partial x^a}(x, \xi) \right) \partial_{\xi_j} + \rho_i^b(x) \frac{\partial H}{\partial \xi_i}(x, \xi) \partial_{x^b}. \quad (14)$$

Another geometrical construction in the skew-algebroid setting is the *complete lift of an algebroid section* (cf. [17, 18]). For every C^1 -section $X = f^i(x) e_i \in \text{Sec}(E)$ we can construct canonically a vector field $d_T^\Pi(X) \in \text{Sec}(TE)$ which in local coordinates reads as

$$d_T^\Pi(X)(x, y) = f^i(x) \rho_i^a(x) \partial_{x^a} + \left(y^i \rho_i^a(x) \frac{\partial f^k}{\partial x^a}(x) + c_{ij}^k(x) y^i f^j(x) \right) \partial_{y^k}. \quad (15)$$

The vector field $d_T^\Pi(X)$ is homogeneous (linear) with respect to y 's.

An E -connection on a vector bundle $\zeta : A \rightarrow M$ is a 'covariant derivative' $\nabla : \text{Sec}(E) \times \text{Sec}(A) \rightarrow \text{Sec}(A)$ such that $\nabla_{fX} Y = f \nabla_X Y$ and

$$\nabla_X(fY) = \rho(X)(f)Y + f \nabla_X Y \quad (16)$$

for any $f \in C^\infty(M)$, $X \in \text{Sec}(E)$, and $Y \in \text{Sec}(A)$. This means that $D = \nabla_X$ is a *quasi-derivation* (called also *derivative endomorphism*, *module derivation*, *covariant differential operator*, etc.; see historical remarks in [24] and references therein) on sections of A with the anchor $\widehat{D} = \rho(X) \in \mathfrak{X}(M)$. The same property has the operator $ad_X^\Pi = [X, \cdot]_\Pi$ associated with any skew algebroid bracket. Actually, a skew algebroid is a skew-symmetric bilinear operation which is a quasi-derivation with respect to each argument.

Note that a quasi-derivation D on sections of A induces canonically the *dual quasi-derivation* D^* on sections of A^* , with the same anchor $\widehat{D} \in \mathfrak{X}(M)$, defined via

$$\langle D^*(\alpha)|Y \rangle = \widehat{D}(\langle \alpha|Y \rangle) - \langle \alpha|D(Y) \rangle. \quad (17)$$

It is easy to see that linear vector fields on A are projectable and correspond to quasi-derivations on sections of A^* .

Theorem 2.4. *There is a one-one correspondence between quasi-derivations D on sections of A^* and linear vector fields χ^D on a vector bundle A over M established by the formula*

$$\chi^D(\iota(\alpha)) = \iota(D(\alpha)). \quad (18)$$

The vector field χ^D is projectable onto the anchor $\widehat{D} \in \mathfrak{X}(M)$.

The following theorem is an abstract version of the complete tangent lift for sections of a skew algebroids as defined in [16, 17, 18].

Theorem 2.5. *With any quasi-derivation D on a vector bundle A over M there is associated a unique linear vector field D^c on A , the complete lift of D , defined as $D^c = \chi^{D^*}$. In local coordinates (x^a, z^i) associated with a local basis f_i of sections of A , the complete lift of the quasi-derivation*

$$D(h^i(x)f_i) = \left(h^i(x)b_i^k(x) + \frac{\partial h^k}{\partial x^a} \sigma^a(x) \right) f_k$$

reads as

$$D^c(x, z) = \sigma^a(x)\partial_{x^a} - z^i b_i^k(x)\partial_{z^k}. \quad (19)$$

Note that $d_T^\Pi(X)$ of a section X of a skew algebroid is defined as $(ad_X^\Pi)^c$ and, like in [16, 17, 18]), the vector field D^c can be characterized also as the unique vector field on A such that, for each section Y of A , the vertical part of D^c with respect to the decomposition of $TA|_{Y(M)}$ into vertical vectors and vectors tangent to the image $Y(M)$ of Y is the negative of the vertical lift, $-D(Y)^v$, of the section $D(Y)$:

$$\forall x \in M \quad (D^c + D(Y)^v)(Y(x)) \in T_{Y(x)}Y(M). \quad (20)$$

Since the complete lifts are linear vector fields, with every time dependent quasi-derivation $D = \{t \mapsto D_t\}$ we can associate a (local) one-parameter family $t \mapsto \exp_t(D)$ of automorphisms of the vector bundle A such that

$$\left. \frac{d}{dt} \right|_{t=t_0} \exp_t(D)(v) = D_{t_0}^c(v), \quad \exp_0(D) = id_A.$$

The trajectories $t \mapsto \exp_t(D)(v)$ of the time-dependent vector field $t \mapsto D_t^c$ on A project onto trajectories of the time-dependent vector field $t \mapsto \widehat{D}_t$ on M .

For connections on line bundles, exactly like in the case of a Lie algebroid, we have the following.

Theorem 2.6. *Let ∇ be an E -connection on a line bundle L and let σ be a nowhere vanishing section of L . Then, there is a section ϕ_σ^∇ of E^* (called the characteristic form of σ) such that, for all $X \in \text{Sec}(E)$,*

$$\nabla_X \sigma = \langle X, \phi_\sigma^\nabla \rangle \sigma. \quad (21)$$

Moreover, if $\sigma' = e^f \cdot \sigma$ for some smooth function $f \in C^\infty(M)$, then $\phi_{\sigma'}^\nabla = \phi_\sigma^\nabla + d^\Pi f$.

In view of the above theorem, the class of ϕ_σ^∇ in the quotient space $[\mathcal{A}^1(E^*)] = \mathcal{A}^1(E^*)/d^\Pi \mathcal{A}^0(E^*)$ does not depend on the choice of the section σ trivializing the bundle L . We call it the *characteristic class* of the connection ∇ and denote, with some abuse of notation, $\text{char}(L)$. If E is a Lie algebroid and ∇ is a representation, $\nabla_X \nabla_{X'} - \nabla_{X'} \nabla_X = \nabla_{[X, X']^\Pi}$, then the characteristic form is closed, $d^\Pi \phi_\sigma^\nabla = 0$, so $\text{char}(L) \in H^1(E)$.

Note that if ∇_i is an E -connection on a line bundle L_i over M , $i = 1, 2$, then $\nabla = \nabla_1 \otimes Id + Id \otimes \nabla_2$ is an E -connection on $L_1 \otimes L_2$ and $\text{char}(L_1 \otimes L_2) = \text{char}(L_1) + \text{char}(L_2)$. This allows us to define $\text{char}(L)$ even when L is not trivializable: as $L \otimes L$ is trivializable, we put $\text{char}(L) = \frac{1}{2} \text{char}(L \otimes L)$. We can also use sections $|\sigma|$ of the ‘absolute value’ bundle $|L| = L/\mathbb{Z}_2$ (cf. [15]). Since $\phi_\sigma^\nabla = \phi_{-\sigma}^\nabla = \phi_{|\sigma|}^\nabla$, and since there is a nowhere-vanishing section $|\sigma_0|$ of $|L|$, we can define $\text{char}(L)$ as the class of $\phi(|\sigma_0|)$, where

$$\nabla_X \sigma_0 = \langle X, \phi_{|\sigma_0|}^\nabla \rangle \sigma_0 \quad (22)$$

for any local representative σ_0 of $|\sigma_0|$.

3 Modular class of a skew algebroid

For a skew algebroid (E, Π) consider the line bundle

$$L^E = \wedge^{\text{top}} E \otimes \wedge^{\text{top}} T^* M.$$

An important observation, analogous to the one in [6], is that there is a canonical line bundle connection ∇^Π on L^E :

$$\nabla_X^\Pi(Y \otimes \mu) = \mathcal{L}_X^\Pi(Y) \otimes \mu + Y \otimes \mathcal{L}_{\rho(X)} \mu, \quad (23)$$

where \mathcal{L} is the standard Lie derivative along a vector field. The corresponding characteristic class $\text{char}(L^E) \in [\mathcal{A}^1(E^*)]$ we call the *modular class* of the skew algebroid E and denote $\text{mod}(E, \Pi)$ (shortly, $\text{mod}(E)$ if Π is fixed). We call the skew algebroid *modular* if its modular class vanishes. The direct check shows that, for standard affine local coordinates (x^a, y^i) in E and for the standard local section σ associated with these local coordinates,

$$\sigma = e_1 \wedge \dots \wedge e_n \otimes dx^1 \wedge \dots \wedge dx^m, \quad (24)$$

the modular form $\phi_\sigma^\Pi := \phi_\sigma^{\nabla^\Pi}$ associated with Π as in (1) reads

$$\phi_\sigma^\Pi = \left(\sum_k c_{ik}^k(x) + \sum_a \frac{\partial \rho_i^a}{\partial x^a}(x) \right) e^i. \quad (25)$$

Exactly like in the case of a Lie algebroid, we get easily the following.

Theorem 3.1. *A skew algebroid (E, Π) is modular if and only if there is a nowhere-vanishing section $|\sigma|$ of $|L^E|$ such that $\mathcal{L}_X^\Pi |\sigma| = 0$ for all $X \in \text{Sec}(E)$.*

We understand, clearly, that $\mathcal{L}_X^\Pi|\sigma|$ is represented locally by $\mathcal{L}_X^\Pi\sigma$ for any local sections σ of L^E representing $|\sigma|$. Note that with any nowhere-vanishing section σ of L^E we can canonically associate a volume form (a nowhere-vanishing top-rank form) $\widetilde{\sigma}$ on E^* . In local coordinates,

$$(f(x)e_1 \wedge \dots \wedge e_n \otimes dx^1 \wedge \dots \wedge dx^m)^\sim = f(x) d\xi_1 \wedge \dots \wedge d\xi_n \wedge dx^1 \wedge \dots \wedge dx^m.$$

Similarly, with a nowhere-vanishing section $|\sigma|$ of $|L^E|$ we associate a nowhere-vanishing density $|\widetilde{\sigma}|$.

The volume form $\widetilde{\sigma}$ (density $|\widetilde{\sigma}|$) is *homogeneous*. More precisely, it is homogeneous of degree $n = \text{rank } E$ with respect to the Euler (Liouville) vector field $\Delta = \xi_i \partial_{\xi_i}$ on E^* :

$$\mathcal{L}_\Delta \widetilde{\sigma} = n \widetilde{\sigma}$$

(for all σ representing $|\sigma|$).

Proposition 3.2. *Any homogeneous volume form (density) on the dual E^* of a skew algebroid E equals $\widetilde{\sigma}$ (resp., $|\widetilde{\sigma}|$) for a nowhere-vanishing section σ of L^E (resp., a nowhere-vanishing section $|\sigma|$ of $|L^E|$).*

Proof. We can assume that L^E is trivializable, so there is a nowhere-vanishing section σ_1 of L^E . If Ω is a volume form, Ω (or $-\Omega$, does not matter) can be written as $\Omega = e^\psi \cdot \widetilde{\sigma}_1$ for some $\psi \in C^\infty(E^*)$. Now, if Ω is homogeneous, then

$$n\Omega = \mathcal{L}_\Delta \Omega = \mathcal{L}_\Delta(e^\psi \cdot \widetilde{\sigma}_1) = e^\psi (\Delta(\psi) \widetilde{\sigma}_1 + n \widetilde{\sigma}_1) = \Delta(\psi) \Omega + n \Omega,$$

so $\Delta(\psi) = 0$. Thus, ψ is homogeneous of degree 0, so it is constant along fibres and therefore is the pull-back $\psi = f \circ \pi$ of a function f on the base. Hence, $\Omega = e^f \widetilde{\sigma}_1$ and the section $\sigma = e^f \sigma_1$ is nowhere-vanishing. \square

Let $(\phi_{|\sigma|}^\Pi)^v$ be the vertical vector field on E^* being the vertical lift of the modular form associated with $|\sigma|$. Let $\text{div}_{|\widetilde{\sigma}|}$ denotes the divergence associated with the density $|\widetilde{\sigma}|$, i.e.

$$\mathcal{L}_{\mathfrak{X}} |\widetilde{\sigma}| = \text{div}_{|\widetilde{\sigma}|}(\mathfrak{X}) |\widetilde{\sigma}|.$$

One can easily prove by direct calculations the following (cf. [41]).

Theorem 3.3. *For each Hamiltonian $H \in C^\infty(E^*)$,*

$$(\phi_{|\sigma|}^\Pi)^v(H) = \text{div}_{|\widetilde{\sigma}|} \mathfrak{X}_H.$$

Among all Hamiltonians we can distinguish those which are of *mechanical type*: the kinetic energy + a potential function. The kinetic energy is associated with a bundle (fiberwise) metric, i.e. a smooth family of scalar products on fibers of E^* (or E). Choosing a local basis of orthonormal sections, we can write in the corresponding local coordinates

$$H(x, \xi) = \frac{1}{2} \sum_i \xi_i^2 + V(x). \quad (26)$$

Since, for any such Hamiltonian,

$$(\phi_{|\sigma|}^\Pi)^v(H) = \left(\sum_k c_{ik}^k(x) + \sum_a \frac{\partial \rho_i^a}{\partial x^a}(x) \right) \frac{\partial H}{\partial \xi_i} = \left(\sum_k c_{ik}^k(x) + \sum_a \frac{\partial \rho_i^a}{\partial x^a}(x) \right) \xi_i, \quad (27)$$

the modular form $\phi_{|\sigma|}^\Pi$ vanishes if and only if $(\phi_{|\sigma|}^\Pi)^v(H) = 0$ for a single mechanical Hamiltonian. We thus get the following (cf. [32, 41]).

Theorem 3.4. *Let E be a skew algebroid. The following are equivalent:*

- (a) *E is unimodular.*
- (b) *There is a homogeneous density on E^* which is invariant with respect to all Hamiltonian vector fields.*
- (c) *There is a homogeneous density on E^* which is invariant with respect to a certain hamiltonian vector field associated with a Hamiltonian of mechanical type.*

Proof. (a) \Leftrightarrow (b) We can assume that L^E is trivializable and work with forms instead of densities. It is clear from Theorem 3.3 that if the skew-algebroid is modular, then there is a homogeneous volume form on E^* invariant with respect to all Hamiltonian vector fields. This is $\tilde{\sigma}$ for a nowhere-vanishing section of L^E whose modular form is 0.

We have also the converse. Indeed, if Ω is a homogeneous volume form, we know that it can be written as $\Omega = \tilde{\sigma}$ for some nowhere-vanishing section σ of L^E . If Ω is invariant with respect to all the Hamiltonian vector fields \mathfrak{X}_H , then $0 = \operatorname{div}_{\tilde{\sigma}} \mathfrak{X}_H = (\phi_{\sigma}^{\Pi})^v(H)$ for all Hamiltonians H . This means $(\phi_{\sigma}^{\Pi})^v = 0$, thus $\phi_{\sigma}^{\Pi} = 0$, so the skew algebroid is modular. Finally, (a) \Leftrightarrow (c) follows immediately from (27). \square

It is *a priori* possible that a non-modular skew algebroid admits a non-homogeneous volume form Ω which is invariant with respect to all Hamiltonian vector fields. Since $\Omega = e^{\psi} \tilde{\sigma}$ for some nowhere-vanishing section σ of L^E , the property that Ω is invariant with respect to all \mathfrak{X}_H reads

$$0 = \operatorname{div}_{\Omega}(\mathfrak{X}_H) = \operatorname{div}_{\tilde{\sigma}}(\mathfrak{X}_H) + \mathfrak{X}_H(\psi) = \left((\phi_{\sigma}^{\Pi})^v - \mathfrak{X}_{\psi} \right) (H).$$

As H is arbitrary, $(\phi_{\sigma}^{\Pi})^v = \mathfrak{X}_{\psi}$. If ψ is a basic function, this implies that ϕ_{σ}^{Π} is exact and the algebroid is modular. It is, however, not excluded that $(\phi_{\sigma}^{\Pi})^v = \mathfrak{X}_{\psi}$ for some function ψ which is not basic. Note that the vertical lift of $d^{\Pi}f$, for a basic functions f , is automatically a Hamiltonian vector field, $(d^{\Pi}f)^v = -\mathfrak{X}_f$. In this way we get the following.

Theorem 3.5. *There is a density on E^* which is invariant with respect to all Hamiltonian vector fields if and only if the vertical lift of a (thus any) modular form is a Hamiltonian vector field.*

Remark 3.6. Skew algebroids appear naturally in nonholonomic mechanics and the modular class in this context has been studied also in [7]. Note also that there is an extensive literature devoted to mechanics on Lie algebroids, the program proposed by Weinstein [40] and developed by Martínez and others (see the survey article [30]). In [12] and [10], in turn, it was developed a framework of geometric mechanics on *general algebroids*. The structure of an algebroid on a bundle $\tau : E \rightarrow M$ allows one to develop the Lagrangian formalism for a given Lagrangian function $L : E \rightarrow \mathbb{R}$. Moreover, if E is an almost Lie algebroid, then the associated Euler-Lagrange equations have a variational interpretation: a curve $\gamma : [t_0, t_1] \rightarrow E$ satisfies the Euler-Lagrange equations if and only if it is an extremal of the action $\mathcal{J}(\gamma) := \int_{t_0}^{t_1} L(\gamma(t))dt$ restricted to those γ 's which are admissible and belong to a fixed E -homotopy class [10].

In [14] it has been shown that if $D \subset E$ is a subbundle (nonholonomic constraint) and L is of mechanical type (that is $L(x, v) = \frac{1}{2}\mu(v, v) - V(x)$, where μ is a bundle metric on E and V is an arbitrary function on the base), then nonholonomically constrained Euler-Lagrange equations associated with D can be obtained as unconstrained Euler-Lagrange equations on the skew-algebroid $(D, \rho|_D, [\cdot, \cdot]_{\Pi}^D := P_D[\cdot, \cdot]_{\Pi})$, where $P_D : E \rightarrow D$ denotes the projection orthogonal w.r.t. μ . In other words, the skew algebroid structure is obtained simply by

projecting the tensor Π on D by means of TD . Note that the skew algebroid bracket $[\cdot, \cdot]_\Pi^D$ needs not to satisfy Jacobi identity even if $[\cdot, \cdot]_E$ does.

Example 3.7. To give a concrete example let us study the Chaplygin sleigh. It is an example of a nonholonomic system on the Lie algebra $\mathfrak{se}(2)$ which describes a rigid body sliding on a plane. The body is supported in three points, two of which slides freely without friction, while the third point is a knife edge. This imposes the constraint of no motion orthogonal to this edge (see [3]).

The configuration space before reduction is the Lie group $G = SE(2)$ of the Euclidean motions of the 2-dimensional plane \mathbb{R}^2 . Elements of the Lie algebra $\mathfrak{se}(2)$ are of the form

$$\hat{\xi} = \begin{pmatrix} 0 & \omega & v_1 \\ -\omega & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix} = v_1 E_1 + v_2 E_2 + \omega E_3,$$

where $[E_3, E_1] = E_2$, $[E_2, E_3] = E_1$, and $[E_1, E_2] = 0$.

The system is described by the purely kinetic Lagrangian function $L : \mathfrak{se}(2) \rightarrow \mathbb{R}$,

$$L(v_1, v_2, \omega) = \frac{1}{2} [(J + m(a^2 + b^2))\omega^2 + mv_1^2 + mv_2^2 - 2bm\omega v_1 - 2am\omega v_2].$$

Here, m and J denote the mass and the moment of inertia of the sleigh relative to the contact point, while (a, b) represents the position of the center of mass with respect to the body frame, determined by placing the origin at the contact point and the first coordinate axis in the direction of the knife axis. Of course, as the Lie algebra $\mathfrak{se}(2)$ is modular, the free dynamics admits an invariant volume for the corresponding Hamiltonian. However, our system is subjected to the nonholonomic constraint determined by the linear subspace

$$D = \{(v_1, v_2, \omega) \in \mathfrak{se}(2) \mid v_2 = 0\} \subset \mathfrak{se}(2).$$

Instead of $\{E_1, E_2, E_3\}$ we take another basis of $\mathfrak{se}(2)$:

$$e_1 = E_3, e_2 = E_1, e_3 = -maE_3 - mabE_1 + (J + ma^2)E_2,$$

adapted to the decomposition $D \oplus D^\perp$; $D = \text{span}\{e_1, e_2\}$ and $D^\perp = \text{span}\{e_3\}$. The induced skew-algebroid structure on D is given by

$$[e_1, e_2]_D = \frac{ma}{J + ma^2}e_1 + \frac{mab}{J + ma^2}e_2.$$

Therefore, the structural constant are $\mathcal{C}_{12}^1 = \frac{ma}{J + ma^2}$ and $\mathcal{C}_{12}^2 = \frac{mab}{J + ma^2}$. The skew algebroid D is in this simple case a Lie algebra, as any skew algebra in dimension 2 is Lie. According to (25),

$$\text{mod}(D) = \frac{mab}{J + ma^2}e^1 - \frac{ma}{J + ma^2}e^2,$$

so $\text{mod}(D) \neq 0$ and the associated Hamiltonian does not admit an invariant volume.

4 Supergeometric description

Our aim now is to express the modular class $\text{mod}(E)$ directly in terms of the degree 1 vector field d^Π on the graded manifold $E[1]$. To this end the sheaf of sections of L^E has to be replaced

by the *Berezinian sheaf* $\text{Ber} = \text{Ber}(E[1])$ on the supermanifold $E[1]$ whose nowhere-vanishing sections are *Berezinian volumes* (cf. [19, 20]). A homogeneous Berezinian volume s defines a divergence div_s of a homogeneous vector field X by the formula (see [25])

$$\mathcal{L}_X^\Pi s = (-1)^{|X||s|} s \cdot \text{div}_s(X).$$

Here $\mathcal{L}_X^\Pi s$ is the Lie derivative of the Berezinian volume understood as a differential operator on \mathcal{A} defined by $\mathcal{L}_X^\Pi s = -(-1)^{|X||s|} s \circ X$. Over a superdomain \mathbf{U} with supercoordinates $\mathbf{u} = (x^1, \dots, x^m, \mathbf{y}^1, \dots, \mathbf{y}^n)$ the (right) $\mathcal{A}(\mathbf{U})$ -module $\text{Ber}(\mathbf{U})$ is generated by

$$s = d^{m|n} \mathbf{u} = dx^1 \wedge \dots \wedge dx^m \otimes \partial_{\mathbf{y}^n} \circ \dots \circ \partial_{\mathbf{y}^1},$$

$$\text{Ber}(\mathbf{U}) = \text{Sec}(\mathbf{U}, \text{Ber}) := d^{m|n} \mathbf{u} \cdot \mathcal{A}(\mathbf{U}),$$

and the corresponding divergence of a homogeneous vector field $X = \sum_a g_a \partial_{x^a} + \sum_i h_i \partial_{\mathbf{y}^i}$ reads (cf. [25])

$$\text{div}_s(X) = \sum_a \frac{\partial g_a}{\partial x^a} - (-1)^{|X|} \sum_i \frac{\partial h_i}{\partial \mathbf{y}^i}, \quad (28)$$

where $|X|$ is the degree of X .

It is easy to see that with any section σ of the line bundle $L^E = \wedge^n E \otimes \wedge^m T^*M$ one can associate a Berezinian section s_σ . In other words, we have an embedding S of the space of sections of L^E into the space of Berezinian sections. In local coordinates this embedding reads

$$S: f(x) e_n \wedge \dots \wedge e_1 \otimes dx^1 \wedge \dots \wedge dx^m \mapsto f(x) dx^1 \wedge \dots \wedge dx^m \otimes \partial_{\mathbf{y}^n} \circ \dots \circ \partial_{\mathbf{y}^1},$$

where e_1, \dots, e_q is a basis of local sections of E and $\partial_{\mathbf{y}^i}$ is the derivation of $\mathcal{A}(E^*)$ being the contraction of a section of $\wedge^\bullet E^*$ with e_i .

Recall that for any nowhere-vanishing section $|\sigma|$ of the bundle $|L^E|$ we can find an open covering $\{U_\alpha\}$ on M and local nowhere-vanishing sections σ_α of $L^E|_{U_\alpha}$ such that $\sigma_\alpha = \pm \sigma_{\alpha'}$ on $U_\alpha \cap U_{\alpha'}$. Hence, $S(\sigma_\alpha) = \pm S(\sigma_{\alpha'})$ over $U_\alpha \cap U_{\alpha'}$. But the divergence div_s does not depend on the sign of s , so the collection $|s| = S(|\sigma|)$ of local nowhere-vanishing Berezinian volumes $S(\sigma_\alpha)$ gives rise to a super(divergence) $\text{div}_{|s|}$ on $E[1]$. Note that $|s|$ can be viewed as a nowhere-vanishing section of the sheaf \mathfrak{D} of *Berezinian 1-densities* defined as the Berezinian sheaf twisted by on orientation sheaf. In the literature the sections of \mathfrak{D} are sometimes referred to as *non-oriented Berezinian sections*. It is easy to see that if $|s'| = f|s|$, with an even nowhere-vanishing function f , is another nowhere-vanishing Berezinian 1-density, then $\text{div}_{|s'|}(X) = \text{div}_{|s|}(X) + X(\ln(f))$.

If we assume now that (E, Π) is a skew algebroid and d^Π is the corresponding vector field on $E[1]$ (see (12)), then, according to (28), it is easy to see in local coordinates that, for the standard local section σ of L^E associated with local coordinates (see (24)),

$$\text{div}_{S(\sigma)}(d^\Pi) = \left(\sum_k c_{ik}^k(x) + \sum_a \frac{\partial \rho_i^a}{\partial x^a}(x) \right) \mathbf{y}^i. \quad (29)$$

The latter superfunction represents clearly the local section ϕ_σ^Π of E^* . Moreover, if $\sigma' = e^f \sigma$ for some function $f \in C^\infty(M)$, then $S(\sigma') = e^f S(\sigma)$ and

$$\text{div}_{S(\sigma')}(d^\Pi) = \text{div}_{S(\sigma)}(d^\Pi) + d^\Pi f$$

that represents the section $\phi_\sigma^\Pi + d^\Pi f$ of E^* . All this works well globally for 1-densities and we get the following.

Theorem 4.1. *Let (E, Π) be a skew algebroid and d^Π be the de Rham vector field on the supermanifold $E[1]$ corresponding to Π . Then, the modular class $\text{mod}(E)$ is represented by the degree-1 function $\text{div}_{|s|}(d^\Pi)$ being the (super)divergence of d^Π with respect to any homogeneous nowhere-vanishing Berezinian 1-density $|s|$ on $E[1]$ (associated canonically with a nowhere-vanishing section $|\sigma|$ of the line-bundle $|L^E|$).*

Remark 4.2. We found that the above interpretation of the modular class in the case of Lie algebroids have been indicated by Kontsevich [22]. Note also that superdivergences have been used in constructing generating operators for Lie algebroids in [25], and in defining modular class for even symplectic graded manifolds in [33].

5 Relative modular class

Let now E_0 be a skew subalgebroid in our skew algebroid $(E, [\cdot, \cdot]_\Pi, \rho)$. This means that (see [11, 31], although it seems that the third condition in [31, Definition 4.3.14] is superfluous):

- (a) E_0 is a vector subbundle in E supported on a submanifold M_0 of M ;
- (b) The anchor $\rho : E \rightarrow TM$ maps E_0 into $TM_0 \subset TM$, so that the bracket $[\cdot, \cdot]_\Pi$ is well defined for sections of E_0 , as its value over M_0 does not depend on extensions of these sections;
- (c) Sections of E_0 are closed with respect to the bracket $[\cdot, \cdot]_\Pi$.

It is easy to see that the restriction of the bracket $[\cdot, \cdot]_\Pi$ to sections of E_0 is a skew algebroid bracket associated with a certain bivector field Π_0 on E_0^* . In other words, the above conditions mean that the tensor Π is tangent to the submanifold $E_{|M_0}^*$ and projectable onto a linear bivector field Π_0 on $E_0^* \simeq E_{|M_0}^* / E_0^\perp$, where $E_0^\perp \subset E_{|M_0}^*$ is the annihilator of E_0 . The association $\Pi \mapsto \Pi_0$ is a simple example of a Poisson reduction. It is also clear that E_0 inherits from E a structure of a skew algebroid that is associated with Π_0 .

All this can be immediately seen in local coordinates in terms of the structure functions $\rho(x)$ and $c(x)$. We can choose affine coordinates

$$(x^\alpha, x^A, y^\iota, y^I) = (x^1, \dots, x^{m_0}, x^{m_0+1}, \dots, x^m, y^1, \dots, y^{n_0}, y^{n_0+1}, \dots, y^n)$$

in a neighbourhood of a fiber of E_0 , so that points of M_0 in M are characterized by $x^A = 0$, and points of E_0 in E by $x^A = 0, y^I = 0$. The fact that E_0 is a subalgebroid means that $\rho_\iota^A(x^a, 0) = 0$ and $c_{\iota\iota'}^K(x^a, 0) = 0$, so that

$$\Pi_0(x^\alpha) = c_{\iota\iota'}^\kappa(x^\alpha, 0) \xi_\kappa \partial_{\xi_\iota} \otimes \partial_{\xi_{\iota'}} + \rho_\iota^\beta(x^\alpha, 0) \partial_{\xi_\iota} \wedge \partial_{x^\beta}.$$

It is also easy to see that subbundles of E (supported over submanifolds) are graded submanifolds of $E[1]$, so that we have the following equivalent descriptions of a skew subalgebroid.

Theorem 5.1. *Let E_0 be a subbundle, supported on $M_0 \subset M$, of a skew algebroid (E, Π) over M (equivalently, a graded submanifold $E_0[1]$ of the N -manifold $\mathcal{M} = E[1]$). The following are equivalent:*

- (a) E_0 is a subalgebroid with a skew algebroid structure induced by a bivector Π_0 on E_0^* .
- (b) The annihilator E_0^\perp is a coisotropic submanifold of the Poisson manifold (E^*, Π) .

- (c) The (super)vector field d^Π is tangent to the submanifold $E_0[1]$ of $E[1]$ and d^{Π_0} is the restriction $d^\Pi|_{E_0[1]}$.
- (d) the canonical inclusion $j_{E_0} : E_0 \hookrightarrow E$ is a morphism of skew algebroids, i.e., the induced restriction map $j_{E_0}^* : \mathcal{A}(E^*) \rightarrow \mathcal{A}(E_0^*)$ intertwines the derivations d^Π and d^{Π_0} ,

$$j_{E_0}^* \circ d^\Pi = d^{\Pi_0} \circ j_{E_0}^* . \quad (30)$$

Proof. (a) \Rightarrow (b) The ideal J of function vanishing on E_0^\perp is generated by basic functions vanishing on M_0 and sections of E extending sections of E_0 . The equations (4) and (5) immediately imply that $\{J, J\}_\Pi = 0$.

(b) \Rightarrow (c) As the ideal \mathbf{J} of functions on $E[1]$ vanishing on $E_0[1]$ is generated by basic functions vanishing on M_0 and functions associated with extensions of sections of E_0^\perp , so locally x^A and \mathbf{y}^I , we get $d^\Pi(\mathbf{J}) \subset \mathbf{J}$ in view of (9), (10), and the fact that $\rho_\iota^A(x^a, 0) = 0$ and $c_{\iota\nu}^K(x^a, 0) = 0$.

(c) \Rightarrow (d) is obvious, and (d) implies that $\rho_\iota^A(x^a, 0) = 0$ and $c_{\iota\nu}^K(x^a, 0) = 0$, so (a). \square

Note that (30) implies that $j_{E_0}^*$ induces a map on classes,

$$j_{E_0}^* : [\mathcal{A}^1(E^*)] \rightarrow [\mathcal{A}^1(E_0^*)]$$

($j_{E_0}^* : H^1(E) \rightarrow H^1(E_0)$ in the case of a Lie algebroid). Let $\text{mod}(E)$ (resp, $\text{mod}(E_0)$) be the modular class of the skew algebroid $(E, [\cdot, \cdot]_\Pi)$ (resp., $(E_0, [\cdot, \cdot]_{\Pi_0})$). It is easy to see that, in general,

$$\text{mod}(E_0) \neq j_{E_0}^*(\text{mod}(E)) .$$

Actually, as easily checked in local coordinates as above and the corresponding sections of L^E ,

$$\begin{aligned} j_{E_0}^*(\phi_\sigma^\Pi)(x^\alpha) &= \sum_{\iota=1}^{n_0} \left(\sum_{\kappa=1}^{n_0} c_{\iota\kappa}^\kappa(x^\alpha, 0) + \sum_{K=n_0+1}^n c_{\iota K}^K(x^\alpha, 0) + \sum_{\beta=1}^{m_0} \frac{\partial \rho_\iota^\beta}{\partial x^\beta}(x^\alpha, 0) \right) e^\iota \\ &= \phi_\sigma^{\Pi_0}(x^\alpha) + \sum_{\iota=1}^{n_0} \sum_{K=n_0+1}^n c_{\iota K}^K(x^\alpha, 0) e^\iota . \end{aligned} \quad (31)$$

The class

$$\text{mod}(E_0; E) = \text{mod}(E_0) - j_{E_0}^*(\text{mod}(E)) ,$$

represented by

$$\phi_\sigma^{\Pi_0}(x^\alpha) - j_{E_0}^*(\phi_\sigma^\Pi)(x^\alpha) = - \sum_K c_{\iota K}^K(x^\alpha, 0) e^\iota , \quad (32)$$

we will call the *relative modular class* of the subalgebroid E_0 in E .

It is easy to see that there is a canonical E_0 -connection $\nabla^{\nu(E_0)}$ in the normal bundle $\nu(E_0) = E|_{M_0}/E_0$ (*linear holonomy connection*):

$$\nabla_X^{\nu(E_0)}([e]) = [[X, e]_\Pi] , \quad (33)$$

(where $[e]$ is the class of a section e of E in $\nu(E_0)$), thus a canonical connection $\nabla^{\nu(E_0)}$ in the line bundle $\wedge^{\text{top}}\nu(E_0)$. In particular, for $\iota = 1, \dots, n_0$ and $J = n_0 + 1, \dots, n$,

$$\nabla_{e_\iota}^{\nu(E_0)}([e_J]) = \sum_{K=n_0+1}^n c_{\iota J}^K(x^\alpha, 0)[e_K] \quad (34)$$

and, for a section $X = f^\iota(x^1, \dots, x^{m_0})e_\iota$ of E_0 ,

$$\nabla_X^{\nu(E_0)}([e_{n_0+1}] \wedge \dots \wedge [e_n]) = \sum_{\iota=1}^{n_0} \sum_{K=n_0+1}^n c_{\iota K}^K(x^\alpha, 0) f^\iota(x^\alpha) [e_{n_0+1}] \wedge \dots \wedge [e_n].$$

This, compared with (32), proves the following.

Theorem 5.2. *The relative class $\text{mod}(E_0; E)$ is the obstruction to the existence of a transversal to E_0 measure invariant under the linear holonomy:*

$$\text{mod}(E_0; E) = -\text{char}(\wedge^{\text{top}}\nu(E_0)) = \text{char}(\wedge^{\text{top}}\nu^*(E_0)). \quad (35)$$

6 Holonomy

Let now $\gamma : I = [0, 1] \rightarrow E$ be a smooth *admissible path* in the skew algebroid E , i.e.,

$$\forall t \in I \quad \rho(\gamma(t)) = \dot{\underline{\gamma}}(t), \quad (36)$$

where $\underline{\gamma} : I \rightarrow M$ is the projection of γ to M , $\underline{\gamma} = \tau \circ \gamma$, and $\dot{\underline{\gamma}} : I \rightarrow TM$ is the tangent prolongation of $\underline{\gamma}$.

Fixing an E -connection ∇ on a vector bundle A over M , we can define the *parallel transport of the fibers of A of along γ* (cf. [4, 8]) as the path $[0, 1] \ni t \mapsto F_t$, $F_t : A_{\underline{\gamma}(0)} \rightarrow A_{\underline{\gamma}(t)}$, of fiber morphisms associated with solutions of the differential equation

$$\nabla_{\gamma(t)} Y(t) = 0, \quad Y(t) \in A_{\underline{\gamma}(t)}, \quad (37)$$

in an obvious way. In local coordinates associated with a choice of a local basis f_1, \dots, f_s of sections of A , (37) reads as

$$\dot{Y}^j(t) + \Gamma_{ik}^j(\underline{\gamma}(t)) \gamma^i(t) Y^k(t) = 0, \quad (38)$$

where

$$\nabla_{e_i}(u^k(x)f_k) = \left(\rho_i^a(x) \frac{\partial u^j}{\partial x^a}(x) + \Gamma_{ik}^j(x) u^k(x) \right) f_j.$$

Note that

$$\gamma^i(t) \rho_i^a(\underline{\gamma}(t)) \frac{\partial Y^j}{\partial x^a}(\underline{\gamma}(t)) = \dot{Y}^j(t)$$

due to admissibility of γ , independently on the extension of Y to a time-dependent section of A . Like in the case of a Lie algebroid, one can easily prove that the parallel transport along γ can be described as the flow $\exp_t(D)$ over $\underline{\gamma}$,

$$F_t(v) = \exp_t(D)(v), \quad v \in A_{\underline{\gamma}(0)}, \quad (39)$$

where $D_t = \nabla_{\gamma(t)}$. Of course, since formally $\nabla_{\gamma(t)}$ is not a globally defined quasi-derivative, one takes ∇_{X_t} instead, where $X = \{t \mapsto X_t\}$ is any time-dependent section of E such that $X_t(\underline{\gamma}(t)) = \gamma(t)$; the r.h.s of (39) does not depend on the choice of X .

Let $\omega = dY^1 \wedge \cdots \wedge dY^s$ be a homogeneous volume form on fibers associated with the chosen local basis of sections. In view of (38), the determinant of the parallel transport, $h(t) = \det(F_t)$, expressed in these coordinates satisfies the differential equation

$$\dot{h}(t) + \sum_{i,j} \Gamma_{ij}^j(\underline{\gamma}(t)) \gamma^i(t) h(t) = 0, \quad (40)$$

thus

$$h(1) = \exp \left(- \int_0^1 \left(\sum_{i,j} \Gamma_{ij}^j(\underline{\gamma}(t)) \gamma^i(t) \right) dt \right). \quad (41)$$

All this, applied to the linear holonomy connection $\nabla^{\nu(E_0)}$ in the normal bundle $A = \nu(E_0) = E|_{M_0}/E_0$ associated with a subalgebroid E_0 of E , gives (see (34))

$$\begin{aligned} h(1) &= \exp \left(\int_0^1 \left(\sum_{\iota=1}^{n_0} \sum_{K=n_0+1}^n c_{\iota K}^K(\underline{\gamma}(t)) \gamma^\iota(t) \right) dt \right) \\ &= \exp \left(\int_0^1 \langle (\phi_\sigma^{\Pi_0} - j_{E_0}^*(\phi_\sigma^\Pi))(\underline{\gamma}(t)), \gamma(t) \rangle dt \right) = \exp \left(\int_\gamma (\phi_\sigma^{\Pi_0} - j_{E_0}^*(\phi_\sigma^\Pi)) \right). \end{aligned} \quad (42)$$

Here, for $\alpha \in \text{Sec}(E^*)$, we denote

$$\int_\gamma \alpha = \int_0^1 \langle \alpha(\underline{\gamma}(t)), \gamma(t) \rangle dt.$$

It is easy to see that, for $\alpha = d^\Pi f$ and γ admissible, we have

$$\int_\gamma d^\Pi f = f(\underline{\gamma}(1)) - f(\underline{\gamma}(0)).$$

In particular, if $\underline{\gamma}$ is a loop, $\underline{\gamma}(0) = \underline{\gamma}(1)$, then $\int_\gamma \alpha$ does not depend on the class of α in $\mathcal{A}^1(E^*)/d^\Pi \mathcal{A}^0(E^*)$. In this case, $\text{hol}(\gamma, E_0) := h(1)$ is well defined independently on the choice of local coordinates and called the *linear holonomy* of γ relative to E_0 . Also the r.h.s. of (42) is well defined independently on the choice of coordinates and can be rewritten in the form $\exp \left(\int_\gamma \text{mod}(E_0; E) \right)$. In this way we get the following generalization of the results of [2, 9] on holonomy of Poisson manifolds and their Poisson submanifolds.

Theorem 6.1. *For any subalgebroid E_0 of a skew algebroid E over M and for any smooth admissible path γ in E covering a loop in M , one has*

$$\text{hol}(\gamma, E_0) = \exp \left(\int_\gamma \text{mod}(E_0; E) \right). \quad (43)$$

7 Modular classes of algebroid relations

Given now two skew algebroids, (E_i, Π_i) over the base M_i , $i = 1, 2$, we can consider the *direct product skew algebroid* $E_1 \times E_2$ over $M_1 \times M_2$ corresponding to the bivector field ' $\Pi_1 + \Pi_2$ ' on $E_1^* \times E_2^*$. Interpreting $\mathcal{A}(E_1^* \times E_2^*)$ as $\mathcal{A}(E_1^*) \otimes \mathcal{A}(E_2^*)$, we can write

$$d^{\Pi_1 \times \Pi_2} = d^{\Pi_1} \otimes 1 + 1 \otimes d^{\Pi_2},$$

or simply, with some abuse of notation, $d^{\Pi_1 \times \Pi_2} = d^{\Pi_1} + d^{\Pi_2}$.

The canonical projections $p_i : E_1 \times E_2 \rightarrow E_i$, $i = 1, 2$, induce the maps $p_i^* : \mathcal{A}(E_i^*) \rightarrow \mathcal{A}(E_1^* \times E_2^*)$ which are algebroid morphisms, i.e. they intertwine the corresponding de Rham derivatives,

$$p_i^* \circ d^{\Pi_i} = (d^{\Pi_1} + d^{\Pi_2}) \circ p_i^*. \quad (44)$$

One can easily prove the following.

Theorem 7.1.

$$\text{mod}(E_1 \times E_2) = p_1^*(\text{mod}(E_1)) + p_2^*(\text{mod}(E_2)). \quad (45)$$

We will write in short, with some abuse of notation,

$$\text{mod}(E_1 \times E_2) = \text{mod}(E_1) + \text{mod}(E_2).$$

The following is generally known in the case of Lie algebroids (cf. [31, Theorem 10.4.9] and [1, 39, 38]).

Theorem 7.2. *Let (E_i, Π_i) be a skew algebroid over a base M_i , $i = 1, 2$, and let $\Phi : E_1 \rightarrow E_2$ be a vector bundle morphism covering a smooth map $\varphi : M_1 \rightarrow M_2$. The following are equivalent.*

- (i) *The graph $\text{graph}(\Phi)$ of Φ is a subalgebroid in the direct product skew algebroid $E_1 \times E_2$.*
- (ii) *The annihilator $\text{graph}(\Phi)^\perp$ is a coisotropic submanifold in the Poisson manifold $(E_1^* \times E_2^*, \Pi_1 \times \Pi_2)$.*
- (iii) *The super vector field $d^{\Pi_1} + d^{\Pi_2}$ is tangent to the submanifold $\text{graph}(\Phi)[1]$ of the graded manifold $(E_1 \times E_2)[1]$.*
- (iv) *The induced map $\Phi^* : \mathcal{A}(E_2) \rightarrow \mathcal{A}(E_1)$ intertwines the de Rham derivatives,*

$$\Phi^* \circ d^{\Pi_2} = d^{\Pi_1} \circ \Phi^*.$$

Proof. It is clear that $\text{graph}(\Phi)$ is a vector subbundle in $E_1 \times E_2$ supported on the graph $\text{graph}(\varphi)$, and we have (i) \Leftrightarrow (ii) \Leftrightarrow (iii) according to Theorem 5.1. Assume now (i). The embedding $j = \text{Id}_{E_1} \times \Phi : E_1 \rightarrow E_1 \times E_2$ is an isomorphism of the vector bundle E_1 onto $\text{graph}(\Phi)$ that identifies the skew algebroid structure on E_1 with that on $\text{graph}(\Phi)$. We have then

$$j^* \circ (d^{\Pi_1} + d^{\Pi_2}) = d^{\Pi_1} \circ j^*. \quad (46)$$

On the other hand, for any $\mu_i \in \mathcal{A}(E_i^*)$,

$$j^*(\mu_1 \otimes 1 + 1 \otimes \mu_2) = \mu_1 + \Phi^*(\mu_2),$$

so that

$$d^{\Pi_1} \mu_1 + \Phi^*(d^{\Pi_2} \mu_2) = d^{\Pi_1} \mu_1 + d_{\Pi_1}(\Phi^*(\mu_2)) \quad (47)$$

and (iv) follows. Conversely, (47) implies (46), thus (i). \square

Definition 7.3. A vector bundle morphism $\Phi : E_1 \rightarrow E_2$ between skew algebroids (E_i, Π_i) , $i = 1, 2$, which satisfies one (thus all) of the above conditions (i)–(iv) we call a *morphism of skew algebroids*. Any subalgebroid in $E_1 \times E_2$ we call a *skew algebroid relation* between E_1 and E_2 .

It follows that skew algebroid morphisms are particular skew algebroid relations and that the latter can be equivalently determined by coisotropic subbundles in $(E_1^* \times E_2^*, \Pi_1 \times \Pi_2)$. For any skew algebroid relation $\mathcal{R} \subset E_1 \times E_2$ supported on a submanifold $M_0 \subset M_1 \times M_2$ and equipped with the induced skew algebroid structure associated with a tensor Π on \mathcal{R}^* , its relative modular class

$$\text{mod}(\mathcal{R}; E_1 \times E_2) = \text{mod}(\mathcal{R}) - j_{\mathcal{R}}^* (p_1^*(\text{mod}(E_1)) + p_2^*(\text{mod}(E_2))) ,$$

where $j_{\mathcal{R}} : \mathcal{R} \rightarrow E_1 \times E_2$ is the canonical embedding, can be simply written as

$$\text{mod}(\mathcal{R}; E_1 \times E_2) = \text{mod}(\mathcal{R}) - j_{\mathcal{R},1}^*(\text{mod}(E_1)) - j_{\mathcal{R},2}^*(\text{mod}(E_2)) ,$$

where $j_{\mathcal{R},i} = p_i \circ j_{\mathcal{R}}$, $i = 1, 2$.

We can, however, define another relative modular class, $\text{mod}(\mathcal{R})$, associated with the algebroid relation \mathcal{R} by comparing the classes of E_1 and E_2 with respect to this relation, namely

$$\text{mod}(\mathcal{R}) = j_{\mathcal{R},1}^*(\text{mod}(E_1)) - j_{\mathcal{R},2}^*(\text{mod}(E_2)) . \quad (48)$$

We will call it the *modular class of the skew algebroid relation* \mathcal{R} . By obvious reasons we will call $\alpha_i \in \mathcal{A}^1(E_i^*)$, $i = 1, 2$, \mathcal{R} -related if

$$j_{\mathcal{R},1}^*(\alpha_1) = j_{\mathcal{R},2}^*(\alpha_2) .$$

One important remark is that the r.h.s., thus $\text{mod}(\mathcal{R})$, has to be understood as an element in

$$\mathcal{A}^1(\mathcal{R}) / (j_{\mathcal{R},1}^*(d^{\Pi_1}(C^\infty(M_1))) + j_{\mathcal{R},2}^*(d^{\Pi_2}(C^\infty(M_2))))$$

and not as an element in $[\mathcal{A}^1(\mathcal{R})] = \mathcal{A}^1(\mathcal{R}) / d^\Pi(C^\infty(M_0))$. This is because, in the latter case, there is *a priori* no guarantee that we can find representatives α_i of $\text{mod}(E_i)$, $i = 1, 2$, which are \mathcal{R} related even when $\text{mod} \mathcal{R}$ is trivial in $[\mathcal{A}^1(\mathcal{R})] = \mathcal{A}^1(\mathcal{R}) / d^\Pi(C^\infty(M_0))$. Moreover, the r.h.s. of (48) makes sense even when \mathcal{R} is not a subalgebroid and just a vector subbundle of $E_1 \times E_2$, i.e. a *linear relation* between skew algebroids.

With this interpretation, the obvious characterization of modularity of the relation is the following.

Proposition 7.4. *The modular class $\text{mod}(\mathcal{R}) \subset E_1 \times E_2$ vanishes if and only if there are nowhere-vanishing densities $|\sigma_i| \in \text{Sec}(|L^{E_i}|)$, $i = 1, 2$, such that $\phi_{|\sigma_1|}^{\Pi_1}$ and $\phi_{|\sigma_2|}^{\Pi_2}$ are \mathcal{R} -related.*

Let $\varphi_{\mathcal{R},i}$ be the projection of M_0 on M_i , $i = 1, 2$. Completely analogously like for Lie algebroids we can consider pull-back bundles $\varphi_{\mathcal{R},1}^!(L^{E_1})$ and $\varphi_{\mathcal{R},2}^!((L^{E_2})^*)$ equipped with the corresponding pull-back connections $j_{\mathcal{R},1}^!(\nabla^{\Pi_1})$ and $j_{\mathcal{R},2}^!(\nabla^{\Pi_2})^*$ (cf. [27, Proposition 1.2]). Also the following is immediate (cf. [27, Proposition 2.1]).

Proposition 7.5. *If $\mathcal{R} \subset E_1 \times E_2$ is a skew algebroid relation, then*

$$\text{mod}(\mathcal{R}) = \text{char} \left((j_{\mathcal{R},1})^! (L^{E_1}) \otimes (j_{\mathcal{R},2})^! (L^{E_2})^* \right) . \quad (49)$$

Moreover, if $\mathcal{R} = \text{graph}(\Phi)$ is the graph of a skew algebroid morphism $\Phi : E_1 \rightarrow E_2$, then

$$\text{mod}(\mathcal{R}) = \text{mod}(E_1) - \Phi^*(\text{mod}(E_2)) .$$

For a skew algebroid morphism $\Phi : E_1 \rightarrow E_2$ we will call $\text{mod}(\Phi) = \text{mod}(E_1) - \Phi^*(\text{mod}(E_2))$ the *modular class of the algebroid morphism* Φ . Note that, in the case when Φ is a Lie algebroid morphism, this coincides with the standard definition (see [26]).

Example 7.6. Let now (M_i, Λ_i) , $i = 1, 2$, be Poisson manifolds and $\varphi : M_1 \rightarrow M_2$ be a Poisson map. This means that the graph of φ is a coisotropic submanifold in $M_1 \times \overline{M_2}$, where $\overline{M_2}$ is the manifold M_2 with the opposite Poisson structure $-\Lambda_2$ [39]. It is well known that any Poisson manifold (M, Λ) gives rise to a Lie algebroid structure on T^*M associated with the linear Poisson tensor $d_T\Lambda$ on TM , the complete tangent lift of Λ (cf. [16, 17]). The direct product of the tangent bundles $TM_1 \times \overline{TM_2}$ carries therefore a linear Poisson structure $\Pi_1 \times (-\Pi_2)$, where $\Pi_i = d_T\Lambda_i$, $i = 1, 2$, which corresponds to a Lie algebroid structure on $T^*M_1 \times \overline{T^*M_2}$. Since $T\text{graph}(\varphi)$ is the graph of $T\varphi$, it is a coisotropic subbundle in $TM_1 \times \overline{TM_2}$, thus corresponds to a Lie algebroid relation $\mathcal{R} \subset T^*M_1 \times \overline{T^*M_2}$, where $\mathcal{R} = (T\text{graph}(\varphi))^\perp$. It is easy to see that the modular class

$$\text{mod}(\mathcal{R}) = j_{\mathcal{R},1}^*(\text{mod}(T^*M_1, \Pi_1)) - j_{\mathcal{R},2}^*(\text{mod}(T^*M_2, -\Pi_2))$$

coincides with $2\text{mod}(\varphi)$, where $\text{mod}\varphi$ is the modular class of the Poisson map φ as defined in [2]. The factor 2 comes from the fact that the Lie algebroid modular class $\text{mod}(T^*M)$ is $2\text{mod}(M)$, where $\text{mod}(M)$ is the modular class of a Poisson manifold (M, Λ) as defined in [41]. Actually, what is used in [2] implicitly is the relation $\mathcal{R}' = (T\varphi)^* \subset TM_1 \times TM_2$, the dual relation of $T\varphi$, embedded in $TM_1 \times TM_2$ by the map $(j_{\mathcal{R},1}, -j_{\mathcal{R},2})$, and the minus sign is canceled by the sign at Π_2 , so the result is the same.

Remark 7.7. Of course, in the above example, instead of $\text{graph}(\varphi)$ we can start with an arbitrary coisotropic submanifold in $M_1 \times \overline{M_2}$ which is a Poisson relation between (M_1, Λ_1) and (M_2, Λ_2) .

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